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of the biology and mathematics of their day, and no one would term the pre-Aristotelian geometry illogical because when it was invented, there was no such thing as logic.

Logic is the articulate development of the determination to be rational, and the progress of exact thought brings with it an increasing comprehension of the fact that the will to be rational is at once its motive power and its goal. The aim toward independence of the subjective does not give us independence, but it keeps us moving in that direction in which independence lies,—for he alone is a slave who is content to be one. The study of scientific and mathematical method may not give us forthwith our goal of a universally accepted code; we may not be able to realize the conception of a time when the way men ought to reason, the way men do reason, and the accepted code will coincide in part and whole. Yet it is no small thing that we are able to appreciate that the march of rational thought lies along this line, that logic is given content by the impulse toward rationality and that he knowledge of this content is to be gained by a study of the process itself. We are at least in possession of the differential coefficient.

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INDEPENDENCE PROOFS AND THE THEORY OF IMPLICATION.

Mathematical or Symbolic Logic claims to present a theory of deduction. It is the thesis of this article that the traditional symbolic logic, represented by the system of Whitehead and Russell's *Principia Mathematica,* fails to give a correct theory of mathematical deduction. It will be shown that the notion of mathematical deduction involved in the recognized methods of proving the independence of sets of postulates is incompatible with the theory of deduction furnished by mathematical logic.

Modern workers on the foundations of mathematics have formulated certain methods for the proof of the independence of a set of assumptions or postulates for a deductive theory. In a set of independent postulates no one of the postulates is deducible from the others. To prove the independence of a set of assumptions one must show, for every assumption of the set, that it cannot be derived as a formal consequence from the other assumptions. Huntington[1] has formulated the method as follows, "The method for establishing the independence of a set of postulates consists in exhibiting, in the case of each postulate, an example of a system which satisfies all the other postulates of the set, but not the one in question." "In this section we establish the independence of the postulates A1 – A4, M1 – M5, by exhibiting, in the case of each of the postulates, a system which satisfies all the other postulates, but not the one for which it is numbered. No one of the postulates, therefore, is deducible from the remaining postulates, for, if it were, then any system which possessed all the other properties would possess this property also, which, as the examples show, is not the case." The method has been formulated by Young[2] in almost the same language. "The second fundamental property sought for in a set of assumptions is independence. By this we mean that none of the assumptions can be derived as a formal logical consequence from the others.... In general, an independence proof is constructed in the following way: Let there be given a set of assumptions of any nature, numbered 1, 2, 3,....*n*, and let it be required to prove that assumption *no.k* is independent of all the others. We must find one concrete representation for which all the assumptions, except *no.k*, are satisfied and for which *no.k* is not true. The exhibition of such a concrete representation constitutes an independence proof of the assumption *k*."

The foregoing quotations from mathematicians indicate precisely the notion of independence. It is clear that the concept of independence involves the nature of logical deduction; hence the method of independence proofs affords a test for a theory of logical deduction.

Now it is the claim of symbolic logic that it furnishes the rules of deduction whereby theorems may be deduced from fundamental assumptions. Thus Mr. Russell[3] states, "Symbolic or Formal Logic —I shall use these terms as synonyms—is the study of the various general types of deduction.... What symbolic logic does investigate is the general rules by which inferences are made, and it requires a classification of relations or propositions only in so far as these general rules introduce particular notions."

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[1] The method of independence proofs may be studied in any of the monographs and articles by Prof. E. V. Huntington. The above quotations are from his monograph on *The Fundamental Laws of Addition and Multiplication in Elementary Algebra*, pp. 4 and 32.

[2] J. W. A. Young, *Fundamental Concepts of Algebra and Geometry*, p. 47.

[3] Bertrand Russell, *The Principles of Mathematics*, pp. 10 and 11.

The *Principia Mathematica*[4] offers a theory of logical deduction. Thus, "The purpose of the present section is to set forth the first stage of the deduction of pure mathematics from its logical foundations. This first stage is necessarily concerned with deduction itself, i. e., with the principles by which conclusions are inferred from premises.... It is in fact the theory of how one proposition can be inferred from another. Now in order that one proposition may be inferred from another, it is necessary that the two should have that relation which makes the one a consequence of the other. When a proposition q is a consequence of a proposition p, we say that p implies q. Thus deduction depends upon the relation of implication, and every deductive system must contain among its premises as many of the properties of implication as are necessary to legitimate the ordinary procedure of deduction. In the present section, certain propositions will be stated as premises, and it will be shown that they are sufficient for all common forms of inference. It will not be shown that they are all necessary, and it is possible that the number of them might be diminished. All that is affirmed concerning the premises is (1) that they are true, (2) that they are sufficient for the theory of deduction, (3) that we do not know how to diminish their number."

Thus it is evident that symbolic logic purports to furnish the principles of logical deduction, by means of which theorems are deduced as logical consequences from fundamental assumptions. Logic thus presents an organon of proof.

What now is the nature of deduction in symbolic logic, in comparison with the deduction of mathematical practice? In the following, I shall show that logical deduction, as implied by the method of independence proofs, is at variance with the theory of deduction, as formulated by symbolic logic. Huntington's assumptions for algebra may be used in illustration. Of the set of independent postulates, let us consider the following two,

A2 $(a+b)+c=a+(b+c)$
M5 $a\times b=b\times a$.

Now symbolic logic offers as a true theorem, the rule that a true proposition is implied by any proposition, i. e., theorem *2.02 of *Principia Mathematica*. Let us apply this rule of inference to Huntington's independent postulates.

2.02 $q \cdot \mathrm{c} \cdot p \, \mathrm{c} \, q$;

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[4] Whitehead and Russell, *Principia Mathematica*. Vol. I, p. 94.

substituting M5 for q in 2.02, we get,

$$a\times b=b\times a\cdot\mathrm{c}:p\cdot\mathrm{c}\cdot a\times b=b\times a.$$

Applying *1.1, the Principle of Inference, we deduce

$$p\cdot\mathrm{c}\cdot a\times b=b\times a.$$

Substituting A2 for p in the above expression, we get,

$$(a+b)+c=a+(b+c)\cdot\mathrm{c}\cdot a\times b=b\times a.$$

Since $(a+b)+c=a+(b+c)$ is one of the postulates of the system, we are able to deduce the other postulate by again applying the Principle of Inference.

Thus by rigorous logical deduction we have succeeded in establishing a relation of implication between two independent postulates, and thus the one postulate is logically deducible from the other. The same proof can be applied to the other members of the set, and hence the set of postulates is not independent.

Now it probably will be maintained that the foregoing proof is incorrect. Huntington's postulates are propositional functions, whereas the above theorem of the *Principia* states implications for propositions. But the implication upon which the defender of the *Principia* will rely is formal implication, and formal implication involves a relation of propositional functions.

The contention of critics will be that the concept of independence applies only to postulates, that is, to propositional functions, and not to propositions. But in reply it may be argued that the concept of independence is applicable to propositions, as well as to functions. It is evident that deductive relations can be set up between propositions. In the case of postulates for geometry, those of Euclid, for example, the deductions may be carried out with the propositions which result when a concrete interpretation is given to the system. Indeed, the first deductions were of that nature. Deduction was discovered with reference to propositions, and the extension of the idea of deduction to propositional functions is a comparatively recent development. Hence we must admit that there may be deductive relations between propositions.

Now the aim of independence proofs, practically, is to show, once and for all, that certain deductive relations cannot be established between postulates. Independence proofs are for the purpose of avoiding further attempts to deduce postulates from other postulates. And the results of this method should be available for the relations between propositions, as well as between functions. By independence proofs we prove the postulates of abstract geometry to be independent. But it is equally of interest to know that the fundamental propositions of Euclid, which result when we give a definite interpretation to the blank forms, are also independent. Otherwise we might seek to reduce the number of the propositions which correspond to the postulates of the abstract theory. This was the aim of Euclid and his followers, who tried to prove the parallel postulate, but without success. But this failure did not prove the absence of deductive relations; independence proofs are required to set the matter at rest. Surely it is relevant to say that in a concrete geometry the parallel postulate is independent of the other postulates, and not that it is independent only in the abstract system. I conclude that it is decidedly of value to be able to speak of the independence of propositions.

Thus far I have argued that the notion of independence is of value with respect to propositions, as well as to functions. In that case it is advisable for the sake of simplicity to employ a notion of independence which can be applied to propositions, as well as to functions. This demand is strengthened by the fact that, using the ordinary methods of mathematicians for the deductions between propositions, no one has yet succeeded in establishing deductive relations between propositions which result when concrete interpretations are given to independent postulates. This creates a presumption that no deductive relations can be established between these propositions, and suggests that when no deductive relations can be established between postulates, they cannot be established between the corresponding propositions. That is, it appears as if the mathematicians used independence in the same sense for propositions, as for functions. If now, we hold that independence has the same meaning for propositions as for functions, an independence proof for functions is an independence proof for the corresponding propositions. The presumption created by the failure to find deductive relations for propositions or functions, is then transformed into definite proof by the establishment of independence proofs for the corresponding functions.

I have said that the failure of mathematicians to find deductive relations between the propositions, which correspond to independent postulates, creates a presumption that independence applies to propositions, as well as to postulates. This is further supported by the fact that deductions of theorems from assumptions proceed, in mathematical practice, by the same methods, whether the assumptions are kept in postulate form or are given a concrete interpretation. Thus consider two assumptions $x\mathrm{R}y \cdot y\mathrm{R}z \cdot \mathrm{c} \cdot x\mathrm{R}z$, and $x\mathrm{R}y \cdot \mathrm{c} \cdot y\mathrm{R}x$. In determining their deductive relations, Huntington does not give a concrete interpretation to the x, y, R; x and y may be numbers or points, R may be "smaller than" or "to the left of," etc. Huntington deduces his theorems in functional form. But suppose he started out with the interpretation of x and y as numbers and R as "smaller than." Then his deductions would proceed in exactly the same way. The main purpose of working with blank forms is not merely to obtain generality, but also to avoid the tendency of the mind to rely upon intuition. But in the latter case the argument for the use of the blank form in deduction is psychological rather than logical. If we are careful, we can deduce with propositions, and avoid the dangers of intuition; and our deductive procedure will be exactly the same as for the blank forms. There is no difference in the machinery of deduction for propositions or propositional functions. If that is the case, one is inclined to the opinion that independence must have the same fundamental meaning for propositions, as for propositional functions.

Let us now recapitulate the argument. I showed that by means of the theorems of the *Principia* it is possible to establish deductive relations between independent postulates. Confronted by the distinction between postulate and proposition, I pointed out that in mathematical practice this distinction is not relevant to the concept of independence. There is reason to believe that from the mathematician's point of view, independence of postulates involves the independence of the corresponding propositions. The attempt to save the *Principia* by this distinction is not adequate. Even if we are unable, by means of the *Principia,* to prove the postulates dependent, we can prove the dependence of the corresponding propositions, which seems to be in conflict with mathematical practice. But even this way of escape seems closed to the defenders of the *Principia,* for it seems possible to argue that we can set up implication relations between the propositional functions, which by means of independence proofs are proved to be independent. The question is whether there is an analogue, for propositional functions, of $q \cdot \mathrm{c} \cdot p \,\mathrm{c}\, q$. If there is such an analogue, then Huntington's independent postulates may be substituted and proved no longer independent. There is evidence to believe that we can establish such an analogue.

Consider the proof of *9.34, which is, $(x) \cdot \phi x \cdot \supset : p \cdot \vee \cdot (x) \cdot \phi x$. We start out with *1.3, $q \cdot \supset \cdot p \vee q$, and substituting ϕx for q we get,

$$\phi x \cdot \supset \cdot p \vee \phi x.$$

Now the next step uses *9.13, by means of which a real variable is turned into an apparent variable. This then implies that the ϕx in the above proposition is a propositional function. Now for our purposes we also substitute ψy for p, and thus get

$$\phi x \cdot \supset \cdot \psi y \vee \phi x.$$

By the same principle one can establish

$$\phi x \cdot \supset \cdot \psi y \supset \phi x.$$

Thus we have the analogue of "A true proposition is implied by any proposition," for propositional functions. By substituting Huntington's independent postulates we seem to be able to come out with implication relations between these postulates. Since they are assumed, we can apply *1.11, the Principle of Inference with respect to functions, and thus deduce one independent postulate from the other.

Now it may be objected, although with no apparent ground, that my argument requires the right to substitute ϕx for q and ψy for p. But the legitimacy of this seems to be even further supported by the results obtained by generalization. Let us begin with

$$\phi x \cdot \supset \cdot \psi y \supset \phi x.$$

By the use of *9.13 and proposition *9.21, which is

$$(x) : \phi x \supset \psi x \cdot \supset : (x) \cdot \phi x \cdot \supset \cdot (x) \cdot \psi x,$$

we can set forth the following steps,

$$\phi x \cdot \supset \cdot \psi y \supset \phi x.$$
$$(x) : \phi x \cdot \supset \cdot \psi y \supset \phi x.$$
$$(x) \cdot \phi x \cdot \supset : (x) : \psi y \cdot \supset \cdot \phi x.$$
$$(x) \cdot \phi x \cdot \supset : \psi y \cdot \supset \cdot (x) \cdot \phi x.$$
$$(y) :\cdot (x) \cdot \phi x \cdot \supset : \psi y \cdot \supset \cdot (x) \cdot \phi x.$$
$$(x) \cdot \phi x \cdot \supset : (y) : \psi y \cdot \supset \cdot (x) \cdot \phi x.$$
$$(x) \cdot \phi x \cdot \supset : (y) \cdot \psi y \cdot \supset \cdot (x) \cdot \phi x.$$

Now on p. 139, Vol. I, of the *Principia*, we are told that $(y) \cdot \phi y$ is the same proposition as $(x) \cdot \phi x$. Hence we may write,

$$(x) \cdot \phi x \cdot \supset : (x) \cdot \psi x \cdot \supset \cdot (x) \cdot \phi x.$$

Now *9 proves that apparent variable propositions may be substituted for p, q, r, in the propositions of the numbers from *1 to *5. Hence the proposition at which we arrived, starting from $\phi x \cdot \mathrm{c} \cdot \psi y \cdot \mathrm{c} \cdot \phi x$, is known to be true by another method. Since we have arrived at a true conclusion by correct reasoning, the starting-point must have been at least probably valid. In other words, the true consequence deduced from $\phi x \cdot \mathrm{c} \cdot \psi y \cdot \mathrm{c} \cdot \phi x$ further supports the right to assert it. This line of argument is further supported by the fact that the halfway stage, $(x) \cdot \phi x \cdot \mathrm{c} : \psi y \cdot \mathrm{c} \cdot (x) \cdot \phi x$, evidently can be asserted. For, as a consequence of *9, the truth of

$$(x) \cdot \phi x \cdot \mathrm{c} : p \cdot \mathrm{c} \cdot (x) \cdot \phi x$$

cannot even be questioned, and using the method of *9.34, we get

$$(x) \cdot \phi x \cdot \mathrm{c} : \psi y \cdot \mathrm{c} \cdot (x) \cdot \phi x.$$

We thus have direct and indirect evidence for the right to assert $\phi x \cdot \mathrm{c} \cdot \psi y \, \mathrm{c} \, \phi x$ and to use it to establish implication relations between propositional functions.

Now there is even another way of approaching the matter. From the doctrine of systematic ambiguity one would be inclined to suspect that the *Principia* would permit the substitution of functions for p, q, r, etc. That this has been done seems clear from a study of the proofs.

Consider the proof of *22.51 $\alpha \times \beta = \beta \times \alpha$.[5]

We start out with *22.33 $x \epsilon \alpha \times \beta \cdot \equiv \cdot x \epsilon \alpha \cdot x \epsilon \beta$.

Now *4.3, which is next used, is $p \cdot q \cdot \equiv \cdot q \cdot p$.

Substituting $x \epsilon \alpha$ for p, and $x \epsilon \beta$ for q,

we get $x \epsilon \alpha \cdot x \epsilon \beta \cdot \equiv \cdot x \epsilon \beta \cdot x \epsilon \alpha$.

Thus the procedure of deduction demands that $x \epsilon \alpha$ and $x \epsilon \beta$ be substituted for p and q.

But $x \epsilon \alpha$ and $x \epsilon \beta$ are propositional functions. Now most of the proofs in the calculus of classes require the substitution of propositional functions like $x \epsilon \alpha$, $x \epsilon \beta$, for p, q, etc. Thus it seems that we may substitute propositional functions for p and q. Hence there seems to be no reason why we may not substitute propositional functions in propositions such as $q \cdot \mathrm{c} \cdot p \, \mathrm{c} \, q$. That is, we may substitute Huntington's independent postulates and thus establish implication relations between them. Then by the application of the Principle of Inference it is an easy matter to deduce one propositional function from the other. Thus despite independence proofs, we are able, by means of the *Principia,* to deduce one independent postulate from another, and we have now seen that the difficulties are not set aside by the distinction between propositional functions and propositions.

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[5] The symbol × is employed as the sign of multiplication, instead of the symbol used in the *Principia*.

The only set of independent postulates is one which consists of only one postulate. As soon as two postulates are assumed as the basis of a deductive system, we are able to deduce, as the first theorem, that they are not independent, no matter whether they are propositions or functions.

The conclusion to be drawn from the preceding discussion is that the theory of deduction of the symbolic logicians is not in agreement with the sort of deduction used by the mathematicians.[6] Postulates proved to be independent by the method of independence proofs can be shown to be dependent by the rules of the *Principia.* Either symbolic logic does not furnish safe rules of deduction for pure mathematics, or else mathematicians, who have used independence proofs to show that assumptions are not deductively related, have been wandering in error.

V. F. LENZEN.

BOOK REVIEWS AND NOTES.

MEDICAL CONTRIBUTIONS TO THE STUDY OF EVOLUTION. By *J. G. Adami, M.D., F.R.S., F.R.C.P.* Pp. xviii, 372. London: Duckworth and Co., 1918. Pris 18s. net.

The first part of this very stimulating book consists of the Croonian Lectures of 1917 delivered before the London Royal College of Physicians on "Adaptation and Disease," and the two other parts, "Heredity and Adaptation" and "On Growth and Overgrowth," are reprints of earlier (1892-1914) papers and addresses by the author which bear upon adaptation and tissue modification. In fact, Dr. Adami found that his earlier work was not known to biologists, and it also seemed useful "to present the conclusions reached, not so much from the point of view of their medical bearing, as from that of their biological significance, in order that both morphologist and physician might observe the direction in which medical research is surely leading us with reference to matters which form the basis of general biology."

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[6] The conclusions of this article agree with the criticisms of the *Principia* by Prof. C. I. Lewis. Professor Lewis's algebra of strict implication is the only deductive theory, of which I know, which is in accord with the kind of deduction used by mathematicians in independence proofs.